

THE BEST CONSTANT IN THE KHINTCHINE INEQUALITY OF THE ORLICZ SPACE L_{ψ_2} FOR EQUIDISTRIBUTED RANDOM VARIABLES ON SPHERES

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ABSTRACT. We compute the best constant in the Khintchine inequality for equidistributed random variables on the N -sphere in the Orlicz space L_{ψ_2} .

1. INTRODUCTION

The classical Khintchine inequality compares the L_p -norm of a sum of Rademacher variables with the ℓ_2 -norm of the coefficients of the sum. The computation of the best possible constants has attracted a lot of interest. For the classical case, Haagerup found the best constants for general $p \in (1, \infty)$ in [1]. Also Khintchine inequalities for different kinds of random variables were investigated, for example rotationally invariant random vectors in [3]. A second variation of the problem changes the underlying space. The Khintchine inequality in Orlicz spaces has been considered in various cases, the first example is a paper by Rodin and Semyonov [7].

Let $q > 0$ and $\psi_q(x) := \exp(x^q) - 1$ for $x \in \mathbb{R}$. By $\|\cdot\|_{\psi_q}$ we denote the norm of the Orlicz space $L_{\psi_q}(\Omega, \Sigma, \mu)$. This is given by

$$\|X\|_{\psi_q} := \inf\{c > 0 \mid \mathbb{E} \left[\psi_q \left(\frac{\|X\|}{c} \right) \right] \leq 1\},$$

for $X \in L_{\psi_q}$. By $\|\cdot\|$ we denote the Euclidean norm. For $q \leq 2$ one can still compare the L_{ψ_q} -norm and the ℓ_2 -norm, see [4]. For $q > 2$, Pisier proved that the Lorentz sequence spaces $\ell_{q', \infty}$ ($1/q + 1/q' = 1$), instead of ℓ_2 come into play, see [6]. This fact was already mentioned by Rodin and Semyonov [7].

Here we compute the best constant for the Orlicz space L_{ψ_2} and equidistributed variables on N -dimensional spheres. We apply the technique from [5]. Peskir reduces the case of the Orlicz space to the classical Khintchine inequality in L_q . The optimality of the constants from L_q carries over to L_{ψ_2} . The same reduction technique can be used for variables on spheres. König and Kwapien computed the optimal constants in [3]. Again the optimality carries over. In this paper we prove the following result.

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Theorem 1.1. *Let $X_j, j = 1, \dots, n$ be an i.i.d. sequence of equidistributed random variables on the N -sphere S^{N-1} . For all $a = a_1, \dots, a_n \in \mathbb{R}$ we have*

$$\left\| \sum_{j=1}^n a_j X_j \right\|_{\psi_2} \leq b(N) \left(\sum_{j=1}^n a_j^2 \right)^{\frac{1}{2}},$$

where the constant $b(N) := \sqrt{\frac{2}{N}} \sqrt{\frac{1}{1 - (\frac{1}{2})^{\frac{1}{N}}}}$ is optimal.

Note that $b(N)$ decreases to $\frac{1}{\sqrt{\ln 2}}$ for $N \rightarrow \infty$. In Section 2 we prove that the inequality is true. Therefore we consider the series expansion of the exponential function. Then we apply the Khintchine inequality from [3]. In Section 3 we show that the constant $b(N)$ can not be smaller. We show that with $Y_n := \sum_{j=1}^n \frac{1}{\sqrt{n}} X_j$ we get asymptotic equality in Theorem 1.1 for $n \rightarrow \infty$.

2. PROOF OF THE INEQUALITY

Let $C > 0$. Applying Beppo-Levi we may interchange the limit and the expected value.

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{\left\| \sum_{j=1}^n a_j X_j \right\|^2}{C^2 \sum_{j=1}^n \|a_j\|^2} \right) \right] \\ &= \mathbb{E} \left[\sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{C^{2k} \left(\sum_{j=1}^n \|a_j\|^2 \right)^k} \left\| \sum_{j=1}^n a_j X_j \right\|^{2k} \right] \\ (2.1) \quad &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{C^{2k} \left(\sum_{j=1}^n \|a_j\|^2 \right)^k} \mathbb{E} \left[\left\| \sum_{j=1}^n a_j X_j \right\|^{2k} \right] \end{aligned}$$

Now we apply König's and Kwapien's Khintchine inequality for variables on the sphere and use the constants for $p = 2k$, which gives $\left(\tilde{b}(2k) \right)^{2k} = \left(\frac{2}{N} \right)^k \left(\frac{\Gamma(k + \frac{N}{2})}{\Gamma(\frac{N}{2})} \right)$, see [3, Theorem 3]. We obtain

$$\mathbb{E} \left[\left\| \sum_{j=1}^n a_j X_j \right\|^{2k} \right] \leq \left(\tilde{b}(2k) \left(\sum_{j=1}^n \|a_j\|^2 \right)^{\frac{1}{2}} \right)^{2k} = \tilde{b}(2k)^{2k} \left(\sum_{j=1}^n \|a_j\|^2 \right)^k.$$

This holds for all $k \in \mathbb{N}$ and therefore for every summand in (2.1). Note that $\tilde{b}(2k)$ does not depend on n .

Therefore we get

$$(2.2) \quad \mathbb{E} \left[\exp \left(\frac{\left\| \sum_{j=1}^n a_j X_j \right\|^2}{C^2 \sum_{j=1}^n \|a_j\|^2} \right) \right] \leq \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{C^{2k}} \left(\frac{2}{N} \right)^k \left(\frac{\Gamma(k + \frac{N}{2})}{\Gamma(\frac{N}{2})} \right).$$

Note that $\Gamma(k + \frac{N}{2}) = \Gamma(\frac{N}{2}) \prod_{l=1}^k (k - l + \frac{N}{2})$.

Consider the function $f(x) := (1 - \frac{2}{N}x)^{-\frac{N}{2}}$. The right-hand side of inequality (2.2) is the Taylor expansion of the function f at the point $x = \frac{1}{C^2}$.

So we get

$$\mathbb{E} \left[\exp \left(\frac{\left\| \sum_{j=1}^n a_j X_j \right\|^2}{C^2 \sum_{j=1}^n \|a_j\|^2} \right) \right] \leq f \left(\frac{1}{C^2} \right).$$

Now let $C := b(N) = \sqrt{\frac{2}{N}} \sqrt{\frac{1}{1 - (\frac{1}{2})^{\frac{2}{N}}}}$. Then $f(\frac{1}{C^2}) = 2$ and this proves that the inequality from Theorem 1.1 holds true.

3. PROOF OF THE OPTIMALITY

In this section let $X_j, j \in \mathbb{N}$ be an i.i.d. family of equidistributed random variables on the sphere S^{N-1} . Denote $Y_n := \sum_{j=1}^n \frac{1}{\sqrt{n}} X_j$.

Lemma 3.1. *Let $C \geq \sqrt{\frac{2}{N}} \sqrt{\frac{1}{1 - (\frac{1}{2})^{\frac{2}{N}}}}$. Then the family of random variables*

$$\left(\exp \left(\frac{\left\| \sum_{j=1}^n \frac{1}{\sqrt{n}} X_j \right\|^2}{C} \right) \right), n \in \mathbb{N}$$

is uniformly integrable.

Proof. According to [2, Theorem 6.19] it suffices to prove that for some $p > 1$,

$$I(p) := \sup_{n \in \mathbb{N}} \mathbb{E} \left[\left(\exp \left(\frac{\|Y_n\|^2}{C} \right) \right)^p \right] < \infty.$$

First note that for a N -dimensional Gaussian variable Z we have $\mathbb{E} [\|X_j\|^{2k}] = 1 \leq \mathbb{E} [\|Z\|^{2k}]$. Using a theorem of Zolotarev [8, Theorem 3] this implies

$$\mathbb{P}(\|Y_n\| > t) \leq \exp(-Nq(t)),$$

where $q(t) = \frac{1}{2}(t^2 - \ln t - 1)$. For large t we have $t^2 - \ln t - 1 > \gamma t^2$ for some γ close to 1, say $\gamma \in (\frac{1}{2}, 1)$.

Therefore we find

$$\begin{aligned} I(p) &= \sup_{n \in \mathbb{N}} \int_0^\infty \mathbb{P} \left(\exp \left(p \frac{\|Y_n\|^2}{C^2} \right) > t \right) dt \\ &= 1 + \sup_{n \in \mathbb{N}} \int_1^\infty \mathbb{P} \left(\|Y_n\| > \frac{C}{\sqrt{p}} \sqrt{\ln(t)} \right) dt \\ &\leq 1 + \int_1^\infty t^{-\frac{N}{2} \frac{C^2 \gamma}{p}} dt. \end{aligned}$$

So we can choose $p \in (1, \frac{N}{2} C^2 \gamma)$ such that the latter integral is finite. \square

Lemma 3.2. *Let Z be a N -dimensional Gaussian variable. Then we have*

$$\|Z\|_{\psi_2} = \frac{\sqrt{2}}{\sqrt{1 - (\frac{1}{2})^{\frac{2}{N}}}}.$$

Proof. Let $C > \sqrt{2}$. We compute

$$\begin{aligned} \mathbb{E} \left[\exp \left(\frac{\|Z\|^2}{C^2} \right) \right] &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp \left(\frac{\|x\|^2}{C^2} \right) \exp \left(-\frac{\|x\|^2}{2} \right) dx \\ &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp \left(-\sum_{j=1}^N x_j^2 \left(\frac{1}{2} - \frac{1}{C^2} \right) \right) dx \\ &= \prod_{j=1}^N \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left(-\frac{1}{2} t^2 \left(\frac{C^2 - 2}{C^2} \right) \right) dt \\ &= \left(\frac{C^2}{C^2 - 2} \right)^{\frac{N}{2}}. \end{aligned}$$

Now we have $\left(\frac{C^2}{C^2 - 2} \right)^{\frac{N}{2}} \leq 2$ if and only if $C \geq \sqrt{\frac{2}{1 - (\frac{1}{2})^{\frac{2}{N}}}}$. This proves the lemma. \square

Lemma 3.3. *Let Z be a N -dimensional Gaussian variable. Then we have*

$$\lim_{n \rightarrow \infty} \left\| \sum_{j=1}^n \frac{1}{\sqrt{n}} X_j \right\|_{\psi_2} = \|Z\|_{\psi_2}.$$

Proof. Assume $\limsup_{n \rightarrow \infty} \|Y_n\|_{\psi_2} > \|Z\|_{\psi_2}$. Then there exists a subsequence $n_k, k \in \mathbb{N}$ and some $\epsilon > 0$ such that

$$\|Y_{n_k}\|_{\psi_2} > \|Z\|_{\psi_2} + \epsilon.$$

According to Lemma 3.1 the family $\left(\exp \left(\frac{\|Y_n\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2 \right), n \in \mathbb{N}$ is uniformly integrable. Also

$$G_n := \exp \left(\frac{\|Y_n\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2 - \exp \left(\frac{\|Z\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2, n \in \mathbb{N}$$

is uniformly integrable. For $M > 0$ we have

$$\int G_n d\mathbb{P} \leq \int_{\{G_n \leq M\}} G_n d\mathbb{P} + \sup_{n \in \mathbb{N}} \int_{\{G_n > M\}} G_n d\mathbb{P}.$$

For every fixed $M > 0$, the first integral tends to 0 for $n \rightarrow \infty$ by the central limit theorem. The second integral tends to 0 for $M \rightarrow \infty$ due to the uniform integrability. Therefore

$$\lim_{n \rightarrow \infty} \int \exp \left(\frac{\|Y_n\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2 d\mathbb{P} = \int \exp \left(\frac{\|Z\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2 d\mathbb{P}.$$

This implies

$$\begin{aligned}
2 &\geq \int \exp\left(\frac{\|Z\|}{\|Z\|_{\psi_2}}\right) d\mathbb{P} \\
&> \int \exp\left(\frac{\|Z\|}{\|Z\|_{\psi_2} + \epsilon}\right) d\mathbb{P} \\
&= \lim_{n \rightarrow \infty} \int \exp\left(\frac{\|Y_n\|}{\|Z\|_{\psi_2} + \epsilon}\right) d\mathbb{P} \\
&= \lim_{k \rightarrow \infty} \int \exp\left(\frac{\|Y_{n_k}\|}{\|Z\|_{\psi_2} + \epsilon}\right) d\mathbb{P} \\
&\geq 2,
\end{aligned}$$

which is a contradiction. Therefore $\limsup_{n \rightarrow \infty} \|Y_n\|_{\psi_2} \leq \|Z\|_{\psi_2}$. In the same way we show $\liminf_{n \rightarrow \infty} \|Y_n\|_{\psi_2} \geq \|Z\|_{\psi_2}$. \square

This finishes the proof of our Theorem.

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REFERENCES

- [1] Uffe Haagerup, *The best constants in the Khintchine inequality*, Stud. Math. **70** (1982), 231–283.
- [2] Achim Klenke, *Probability theory. A comprehensive course*, London: Springer, 2008.
- [3] Hermann König and Stanisław Kwapień, *Best Khintchine type inequalities for sums of independent, rotationally invariant random vectors*, Positivity **5** (2001), no. 2, 115–152.
- [4] Michel Ledoux and Michel Talagrand, *Probability in Banach spaces: Isoperimetry and processes*, Berlin: Springer, 1991.
- [5] Goran Peškiri, *Best constants in Kahane-Khintchine inequalities in Orlicz spaces*, J. Multivariate Anal. **45** (1993), no. 2, 183–216.
- [6] Gilles Pisier, *De nouvelles caractérisations des ensembles de sidon*, Adv. Math., Suppl. Stud. **7B** (1981), 685–726.
- [7] V.A. Rodin and E.M. Semyonov, *Rademacher series in symmetric spaces*, Anal. Math. **1** (1975), 207–222.
- [8] V. M. Zolotarev, *Some remarks on multidimensional Bernstein-Kolmogorov-type inequalities*, Theory Probab. Appl. **13** (1968), 281–286, Translation by B. Seckler.

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